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ASYMPTOTIC THEORY OF RIGID PLASTIC SHELLS

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An asymptotic theory of limit shell equilibrium is constructed for rigid-plastic media. The variational approach permits investigation of the limit equilibrium without relying on an analysis of the state of stress. Hence, there is no necessity for the traditional division of the states of the shells into the bending and membrane states. The different states of shells are classified according to the degree of approximation to the exact three-dimensional formulation of the problem. The first figure of the asymptotic expansion is introduced for the kinematic factor and conditions are mentioned when this term of the expansion differs from zero. The asymptotic accuracy of the shell approximation is proved.

1. Formulation of the problem. Let us consider a three-dimensional volume T_h, defined by the relationships

$$\mathbf{r} = \rho(\xi, \eta) + \zeta h \mathbf{n}, \quad \mathbf{n} = (\rho_\xi \times \rho_\eta) \mid \rho_\xi \times \rho_\eta \mid^{-1}, \quad \zeta_1(\xi, \eta) \leqslant \zeta \leqslant \zeta_2(\xi, \eta)$$

$$\zeta_1 < \zeta_2, \quad \mathbf{r} = (x, y, z), \quad (\xi, \eta) \in D, \quad \rho_\xi = \partial\rho/\partial\xi, \quad \rho_\eta = \partial\rho/\partial\eta$$

Here $\rho(\xi, \eta)$ gives the surface in the three-dimensional space. It is assumed that the coordinate lines $\xi = \text{const}, \eta = \text{const}$ thereon are lines of curvature.
Let the volume T_h contain an incompressible rigid-plastic medium with the dissipative potential $\varphi_h(x, y, z, e_{ij})$ [1]. Let us assume that the potential φ_h has the following form in curvilinear coordinates ξ, η, ζ

$$\varphi_h = \varphi(\xi, \eta, \zeta, e_{\xi\xi}, e_{\eta\eta}, e_{\xi\eta}, e_{\zeta\zeta}, e_{\xi\zeta}, e_{\eta\zeta}) \tag{1.1}$$

Here $e = (e_{\xi\xi}, e_{\eta\eta}, e_{\xi\eta}, e_{\zeta\zeta}, e_{\xi\zeta}, e_{\eta\zeta})$ is the strain rate tensor. If $\mathbf{u} = (u, v, w)$ is the vector velocity field in projections on ξ, η, ζ respectively, then

$$e_{\xi\xi} = \frac{1}{H_\xi}\frac{\partial u}{\partial \xi} + \frac{v}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial \eta} - \frac{K_\xi w}{1 - K_\xi h\zeta} \tag{1.2}$$

$$e_{\eta\eta} = \frac{1}{H_\eta}\frac{\partial v}{\partial \eta} + \frac{u}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi} - \frac{K_\eta w}{1 - K_\eta h\zeta}$$

$$e_{\xi\eta} = \frac{1}{2}\left[\frac{H_\xi}{H_\eta}\frac{\partial}{\partial \eta}\left(\frac{u}{H_\xi}\right) + \frac{H_\eta}{H_\xi}\frac{\partial}{\partial \xi}\left(\frac{v}{H_\eta}\right)\right]$$

$$e_{\xi\zeta} = \frac{1}{2}\left[\frac{1}{H_\xi}\frac{\partial w}{\partial \xi} + \frac{H_\xi}{h}\frac{\partial}{\partial \zeta}\left(\frac{u}{H_\xi}\right)\right]$$

$$e_{\eta\zeta} = \frac{1}{2}\left[\frac{1}{H_\eta}\frac{\partial w}{\partial \eta} + \frac{H_\eta}{h}\frac{\partial}{\partial \zeta}\left(\frac{v}{H_\eta}\right)\right]$$

$$e_{\zeta\zeta} = \frac{1}{h}\frac{\partial w}{\partial \zeta}, \quad H_\xi = |\rho_\xi(1 - k_\xi h\zeta)|, \quad H_\eta = |\rho_\eta(1 - K_\eta h\zeta)|$$

Here K_ξ, K_η are the principal surface curvature in the directions ξ, η, respectively. The potential φ is assumed to be a convex function of the first power of the homogeneity relative to the tensor e, , i.e.,

$$\varphi(\xi, \eta, \zeta, \lambda e) = \lambda\varphi(\xi, \eta, \zeta, e), \quad \lambda \geqslant 0$$

and such that there exists a positive β, for which

$$\beta I_2 \leqslant \varphi, \quad I_2 = (e_{\xi\xi}^2 + e_{\eta\eta}^2 + e_{\zeta\zeta}^2 + 2e_{\xi\eta}^2 + 2e_{\xi\zeta}^2 + 2e_{\eta\zeta}^2)^{1/2}$$

Moreover, we assume φ to be a smooth function of its arguments for $I_2 > 0$. In the case of an isotropic medium, it can be shown that φ satisfies the relationship $\varphi(\xi, \eta, \zeta, e) \geqslant \varphi(\xi, \eta, \zeta, \varepsilon)$, where ε is the tensor e, at which $e_{\xi\zeta} = e_{\eta\zeta} = 0$. The velocity fields on which the tensor e, is defined are assumed to be solenoids

$$e_{\xi\xi} + e_{\eta\eta} + e_{\zeta\zeta} = 0 \tag{1.3}$$

and to satisfy certain boundary conditions.
Let a system of forces F_h:(f $(\xi, \eta, \zeta), h\mathbf{f}_i(\xi, \eta)), i = 1.2$ act in a medium enclosed in the volume $T: (\xi, \eta) \in D, \zeta_1 \leqslant \zeta \leqslant \zeta_2$. Here $\mathbf{f}(\xi, \eta, \zeta)$ is the density of the volume forces acting in the domain T, and $\mathbf{f}_i(\xi, \eta)$ is the density of the surface forces concentrated on the surface $\zeta = \zeta_i(\xi, \eta), i = 1, 2$.
One of the most important quantities in the theory of rigid-plastic media is the kinematic factor for the system of forces F_h, i.e., the number c_h, defined by the formula

$$\frac{1}{c_h} = \sup_{\mathbf{u}, \operatorname{div}\mathbf{u}=0}\left\{\left[\int_T \mathbf{fu}\Delta d\xi\, d\eta\, d\zeta + \sum_{i=1}^{2}\int_{S_i}\mathbf{f}_i\mathbf{u}\Delta_i d\xi\, d\eta\right]\left[\int_T \varphi\Delta d\xi\, d\eta\, d\zeta\right]^{-1}\right\} \tag{1.4}$$

$$\Delta = H_\xi H_\eta, \quad S_i : \zeta = \zeta_i(\xi, \eta), \quad \Delta_i = |\mathbf{r}_\xi' \times \mathbf{r}_\eta'|\big|_{\zeta=\zeta_i}, \quad i = 1, 2$$

It will be shown that

$$c_h = c_0 + \alpha(h), \quad \alpha(h) \to 0 \text{ for } h \to 0 \tag{1.5}$$

The number c_0 is called the shell approximation of the kinematic factor and a formula will be obtained for it, which is substantially simpler as compared with (1.4).

2. Shell approximations. Limits of applicability of the shell approximation. Let us introduce the shell approximation by introducing heuristic hypotheses. The foundation for this approximation will be given below.

Let $\mathbf{u}_h$ be a field on which the upper bound in (1.4) is achieved, and let these fields converge to $\mathbf{u}_0$ as $h \to 0$. It follows from the kind of components of the tensor e that $\mathbf{u}_0 = (u_0(\xi, \eta),\ v_0(\xi, \eta),\ w_0(\xi, \eta))$. Therefore, for small h the field $\mathbf{u}_h$ has the form

$$\mathbf{u}_h = \mathbf{u}_0 + h\mathbf{u}_1(\xi, \eta, \zeta, h) \tag{2.1}$$

where the components $_1(\xi, \eta, \zeta, h)$ are bounded in some sense. It follows from the form of the field (2.1) that a change in $\mathbf{u}_1$, on the order of one will cause a change on the order of h in the components $e_{\xi\xi}, e_{\eta\eta}, e_{\xi\eta}$ and on the order of one in the components $e_{\xi\zeta}, e_{\eta\zeta}$. Hence, by selecting the field $\mathbf{u}_1$ it is possible to try to diminish the value of φ in (1.4) if possible. We therefore arrive at the shell potential

$$\psi(\xi, \eta, \zeta, \varepsilon) = \psi(\xi, \eta, \zeta, e_{\xi\xi}, e_{\eta\eta}, e_{\xi\eta}, -e_{\xi\xi} - e_{\eta\eta}) = \min_{e_{\xi\zeta}, e_{\eta\zeta}} \varphi(\xi, \eta, \zeta, e)$$

Neglecting quantities approaching zero in Δ as $h \to 0$, and considering the upper bound in (1.4) on the fields $\mathbf{u} = \mathbf{u}(\xi, \eta)$, we arrive at the following expression for c_0:

$$\frac{1}{c_0} = \sup\left\{\int_D [Pu + Qv + Rw]\, d\xi\, d\eta \left[\int_D \psi_0(\xi, \eta, \varepsilon_0)\, d\xi\, d\eta\right]^{-1}\right\} \tag{2.2}$$

$$P = \left(f_\xi^1 + f_\xi^2 + \int_{\zeta_1}^{\zeta_2} f_\xi d\zeta\right)|\boldsymbol{\rho}_\xi||\boldsymbol{\rho}_\eta|, \quad Q = \left(f_\eta^1 + f_\eta^2 + \int_{\zeta_1}^{\zeta_2} f_\eta d\zeta\right)|\boldsymbol{\rho}_\xi||\boldsymbol{\rho}_\eta|$$

$$R = \left(f_\zeta^1 + f_\zeta^2 + \int_{\zeta_1}^{\zeta_2} f_\zeta d\zeta\right)|\boldsymbol{\rho}_\xi||\boldsymbol{\rho}_\eta|, \quad \mathbf{f} = (f_\xi, f_\eta, f_\zeta)$$

$$\mathbf{f}_i = (f_\xi^{\,i}, f_\eta^{\,i}, f_\zeta^{\,i}), \quad i = 1, 2, \quad \varepsilon_0 = \varepsilon\,|_h = 0$$

$$\psi_0(\xi, \eta, \varepsilon) = \int_{\zeta_1}^{\zeta_2} \psi(\xi, \eta, \zeta, \varepsilon)\, d\zeta\, |\rho_\xi||\rho_\eta|$$

No incompressibility condition has been imposed in (2.2) in determining the quantity c_0 on the admissible field u since this condition can be satisfied because of the selection of $w_1(\xi, \eta, \zeta, h)$. The shell approximation introduced has meaning only for $c_0 \neq 0$.

Let us examine the case when $c_0 = 0$. This holds, say, when the vector (P, Q, R) is not orthogonal to at least one solution of the system of equations $\varepsilon_0(\mathbf{u}) = 0$. The system of equations $\varepsilon_0(\mathbf{u}) = 0$ (three differential equations in three unknown functions) corresponding to an unclosed surface $\mathbf{r} = \boldsymbol{\rho}(\xi, \eta)$, if additional boundary conditions are not imposed for it, has an infinity of linearly independent solutions (in contrast to the over-determined system $e = 0$, which has only the motion T, as a solid as the solution). Therefore, in order for there to be $c_0 \neq 0$, it is necessary to assume that the vector (P, Q, R) is orthogonal to all $\mathbf{u}$, the solution of the system $\varepsilon_0(\mathbf{u}) = 0$. In addition, compliance with an infinity of orthogonality conditions is a very rigid constraint on the class of possible external forces, the formulations of the problem in stresses (such problems are considered in [2], for example) are physically incorrect since an arbitrarily small perturbation of the external forces can violate some of the orthogonality conditions and c_0 thereby vanishes.

3. Computation of the kinematic coefficient of the shell approximation. The problem of finding the kinematic factor c_0 is closely related to the investigation of the properties of the following system:

$$\frac{1}{|\rho_\xi|}\frac{\partial u}{\partial \xi} + \frac{v}{|\rho_\xi||\rho_\eta|}\frac{\partial |\rho_\xi|}{\partial \eta} - K_\xi w = \varepsilon_{\xi\xi}^{\circ} \tag{3.1}$$

$$\frac{1}{|\rho_\eta|}\frac{\partial v}{\partial \eta} + \frac{u}{|\rho_\xi||\rho_\eta|}\frac{\partial |\rho_\eta|}{\partial \xi} - K_\eta w = \varepsilon_{\eta\eta}^{\circ}$$

$$\frac{1}{2}\left[\frac{|\rho_\xi|}{|\rho_\eta|}\frac{\partial}{\partial \eta}\left(\frac{u}{|\rho_\xi|}\right) + \frac{|\rho_\eta|}{|\rho_\xi|}\frac{\partial}{\partial \xi}\left(\frac{v}{|\rho_\eta|}\right)\right] = \varepsilon_{\xi\eta}^{\circ}$$

The theory of boundary value problems and methods of solving this system are developed in [3]. The geometric aspect of the investigation of the system (3.1) is contained in [4].

Let a certain problem be posed for the system (3.1), which is uniquely solvable, i.e. let us assume that there exists an inverse operator

$$\mathbf{u} = G\boldsymbol{\varepsilon}_0, \quad \mathbf{u} = (u, v, w), \quad \boldsymbol{\varepsilon}_0 = (\varepsilon_{\xi\xi}^{\circ}, \varepsilon_{\eta\eta}^{\circ}, \varepsilon_{\xi\eta}^{\circ})$$

with respect to which it is assumed that the conjugate operator G^*

$$\int_D \mathbf{P}(G\boldsymbol{\varepsilon}_0)\, d\xi\, d\eta = \int_D (G^*\mathbf{P})\boldsymbol{\varepsilon}_0 d\xi\, d\eta, \quad \mathbf{P} = (P, Q, R)$$

is a bounded operator acting from the space of sufficiently smooth vector fields $\boldsymbol{\varepsilon}_0$ into the space of continuous vector fields $\mathbf{u}$. Problems in which an operator bounded in the mentioned sense and conjugate to the inverse operator exists for the system (3.1) are called algebraicized ones.

Using the operator G^*, let us rewrite (2.2) in the form

$$\frac{1}{c_0} = \sup_{\boldsymbol{\varepsilon}_0(\xi,\eta)} \left\{\int_D (G^*\mathbf{P})\boldsymbol{\varepsilon}_0 d\xi\, d\eta \left[\int_D \psi_0(\xi, \eta, \boldsymbol{\varepsilon}_0)\, d\xi\, d\eta\right]^{-1}\right\} = \sup_{\xi,\eta,\boldsymbol{\varepsilon}_0} \frac{(G^*\mathbf{P}(\xi,\eta))\boldsymbol{\varepsilon}_0}{\psi_0(\xi,\eta,\boldsymbol{\varepsilon}_0)} \tag{3.2}$$

The theorem on the norm of a linear continuous functional in $L_1(D)$ is used in obtaining (3.2).

Therefore, the question of finding the kinematic factor reduces to an algebraic problem to find the upper bound of a function of five variables $(\xi,\ \eta,\ \varepsilon_{\xi\xi}^{\circ},\ \varepsilon_{\eta\eta}^{\circ},\ \varepsilon_{\xi\eta}^{\circ})$. Let $\xi^*,\ \eta^*,\ \varepsilon_0^*$ be values of the variables for which the upper bound (maximum) is reached in (3.2). Then the extremal field $\varepsilon_0^*(\xi,\eta)$ has the form $\varepsilon_0^*(\xi,\eta) = \varepsilon_0^*\delta(\xi-\xi^*,\ \eta-\eta^*)$. The extremal field $\mathbf{u}^*$. on which the upper bound in (2.2) is reached has the form $\mathbf{u}^* = G\varepsilon_0^*(\xi,\eta)$. Let us note that in the case of algebraicized problems, there is a point $\rho(\xi^*,\eta^*)$, on the surface $\mathbf{r} = \rho(\xi,\eta)$ which determines the nature of shell failure. Let us note also that when ψ_0 is a smooth function of the variables ε_0, the extremal vector ε_0^* is determined from the algebraic equations

$$\frac{F_{\xi\xi}}{\partial\psi_0/\partial\overset{\circ}{\varepsilon}_{\xi\xi}} = \frac{F_{\eta\eta}}{\partial\psi_0/\partial\overset{\circ}{\varepsilon}_{\eta\eta}} = \frac{F_{\xi\eta}}{\partial\psi_0/\partial\overset{\circ}{\varepsilon}_{\xi\eta}} \qquad (3.3)$$

$$G^*P = (F_{\xi\xi},\ F_{\eta\eta},\ F_{\xi\eta})$$

For example, in the case of the Mises potential

$$\psi_0 = a(\xi,\eta)[(\overset{\circ}{\varepsilon}_{\xi\xi})^2 + (\overset{\circ}{\varepsilon}_{\eta\eta})^2 + (\overset{\circ}{\varepsilon}_{\xi\xi} + \overset{\circ}{\varepsilon}_{\eta\eta})^2 + 2(\overset{\circ}{\varepsilon}_{\xi\eta})^2]^{1/2}$$

the system (3.3) is solved easily and the following formula is obtained for c_0

$$\frac{1}{c_0} = \max_{\xi\eta}\left\{\frac{2}{\sqrt{6}\,a(\xi,\eta)}\left[F_{\xi\xi}^2 + F_{\eta\eta}^2 - F_{\xi\xi}F_{\eta\eta} + \frac{3}{4}F_{\xi\eta}^2\right]^{1/2}\right\} \qquad (3.4)$$

4. Examples. Calculation of the quantity c_0 reduces to finding the operator G^*. . An effective construction of the operator G^* can be performed, for example, in the case of second order surfaces of positive Gaussian curvature. In this case, the system (3.1) reduces to a Cauchy-Riemann system.

In the case of shells of revolution, the operator G^* can be found by separation of variables, which results in the solution of an infinite system of linear ordinary differential equations. However, if we represent the external load vector P in the form of a polynomial

$$\mathbf{P}(\xi,\eta) = \sum_{k=-N}^{k=N} \mathbf{P}_k(\xi)\,e^{ik\eta}$$

($\xi = const$ is a parallel on the surface of revolution), then the problem of calculating c_0 reduces to integrating a finite system of equations.

Let us examine specific problems.

Example 1. Let us examine the surface of a torus

$$x = (b + a\sin(\xi/a))\cos\eta, \quad y = (b + a\sin(\xi/a))\sin\eta, \quad z = a\cos(\xi/a)$$

In this case, in order for c_0 to differ from zero, the condition

$$\int_0^{\pi a}\left[P\sin\left(\frac{\xi}{a}\right) - R\cos\left(\frac{\xi}{a}\right)\right]d\xi = 0 \qquad (4.1)$$

must be imposed on the field P, in addition to the condition of orthogonality

of the torus motions as a solid.
Condition (4.1) is a result of the fact that the system (3.1) for a torus with $\varepsilon_0 = 0$, allows a solution of the form

$$v=0, \quad u = \begin{cases} C_1 \sin(\xi / a), & 0 \leqslant \xi \leqslant \pi a \\ C_2 \sin(\xi / a), & \pi a \leqslant \xi \leqslant 2\pi a \end{cases}, \quad w = \begin{cases} -C_1 \cos(\xi / a), & 0 \leqslant \xi \leqslant \pi a \\ -C_2 \cos(\xi / a), & \pi a \leqslant \xi \leqslant 2\pi a \end{cases}$$

in addition to the ordinary solutions corresponding to the motion of the torus as a solid. This solution is a nontrivial infinitesimal flexure of the torus. For instance, the field

$$P = 0, \quad Q = 0, \quad R = p\,(b + a \sin(\xi / a))$$

corresponding to the case when a toroidal shell is subjected to uniform internal pressure with the density $f_\zeta^1 = p$, satisfies the condition (4.1). Under the condition that $\Delta\zeta = \zeta_2 - \zeta_1 = \text{const}$, we have from (3.4) for c_0

$$c_0 = \frac{\tau_0 \sqrt{6}\, \Delta\zeta\, (b - a)}{pa \sqrt{3b^2 - 3ab + a^2}}$$

The failure points (ξ^*, η^*) are hence the following $\xi^* = 3\pi a / 2, \xi^* = \pi a / 2, 0 \leqslant \eta^* \leqslant 2\pi$.
Let us consider the problem of the kinematic factor for a torus rotating around the axis at the angular velocity ω. In this case the centrifugal forces $\mathbf{P}$ act on the torus. Let γ be the volume density of the torus material. Then

$$Q = 0, \quad \Delta\zeta = \text{const},$$
$$P \sin(\xi / a) = R \cos(\xi / a) = A$$
$$A = \gamma\omega^2 (b + a \sin(\xi / a)) \cos(\xi / a) \sin(\xi / a)$$

From (3.4) we find

$$c_0 = \frac{\tau_0}{\omega^2\gamma\,(b + a)} \sqrt{\frac{3}{2}}$$

Example 2. Let us consider a conic surface $\mathbf{r} = \xi\, \mathbf{r}_0(\eta)$, where ξ is the length of a segment of the cone generator measured from its apex, $\mathbf{r}_0(\eta)$ is the line of intersection of the cone with a unit sphere whose center is at the cone apex, and η is the arclength along this line. In the case of surfaces of zero Gaussian curvature the system (3.1) is solved by quadratures [3], and c_0 is found from (3.4). For example, let it be given the circular cone

$$\mathbf{r}_0(\eta) = (\cos\eta \cos\theta, \ \sin\eta \cos\theta, \ -\sin\theta), \quad 0 \leqslant \xi \leqslant l$$

Here $(\pi / 2) - \theta$ is the cone half-angle. We examine an external force field of the following kind: $P = 0$, $Q = 0$, $R = R(\eta)$. We then have from (3.4)

$$\frac{1}{c_0} = \max_{\xi,\eta} \frac{2\xi}{a(\xi,\eta)\sqrt{6}} \left\{ \left(R \operatorname{ctg}\theta - \frac{1}{2\cos^2\theta}\frac{\partial^2 R}{\partial\eta^2}\right)^2 + \frac{3}{4}\left(\frac{1}{\sin\theta}\frac{\partial R}{\partial \eta}\right)^2\right\}^{1/2}$$

Example 3. Let us consider a cylindrical surface. In this case

$$\rho(\xi,\eta) = \mathbf{r}_0(\eta) + \mathbf{v}\xi, \quad \mathbf{v} = (0, 0, 1), \quad \mathbf{r}_0\mathbf{v} = 0, \quad 0 \leqslant \xi \leqslant L$$
$$|\mathbf{r}_0'| = 1, \quad K_\xi = 0, \quad K_\eta = K(\eta), \quad 0 \leqslant \eta \leqslant \eta_0$$

Let us consider some specific problems of determining c_0 for cylindrical shells. We assume that $P = 0$, $Q = 0$, $R = R(\xi)$. Then

$$F_{\xi\xi} = -\Pi(\xi)\left(\frac{1}{k(\eta)}\right)'', \quad F_{\eta\eta} = -\Pi''(\xi)\frac{1}{k(\eta)} \tag{4.2}$$
$$F_{\xi\eta} = 2\Pi'(\xi)\left(\frac{1}{k(\eta)}\right)', \quad \Pi(\xi) = \int_\xi^L R(\tau)(\tau-\xi)\,d\tau$$

Substituing (4.2) into (3.4), we find the formula for c_0.
Let us consider the problem of the stiffness of the covering of a cylindrical shell under its own weight. Let $\mathbf{r}_0(\eta) = (x(\eta), y(\eta), 0)$. Then

$$P = 0, \quad Q = -\gamma g y', \quad R = -\gamma g y'' / k(\eta)$$

In this case (3.4) takes the form

$$\frac{1}{c_0} = \frac{2\gamma g}{\sqrt{6}} \max_{\xi,\eta}\left(\frac{1}{a(\xi,\eta)}\left\{\frac{(L-\xi)^4}{4}\left[\frac{\partial^2}{\partial\eta^2}\left(\frac{y''}{k^2}-y\right)\right]^2 + \left(\frac{y''}{k^2}\right)^2 - \frac{(L-\xi)^2}{2}\frac{y''}{k^2}\frac{\partial^2}{\partial\eta^2}\left(\frac{y''}{k^2}-y\right) + 3(L-\xi)^2\left[\frac{\partial}{\partial\eta}\left(\frac{y''}{k^2}-y\right)\right]^2\right\}^{1/2}\right) \tag{4.3}$$

The shell remains stiff if $c_0 > 1$. Let the coating of constant thickness $\Delta\zeta$ be a part of a circular cylinder of radius l, $0 \leqslant \eta \leqslant \pi l$. Then $k = -1/l$, $x = l\cos(\eta/l)$, $y = l\sin(\eta/l)$. We find from (4.3)

$$\frac{1}{c_0} = \frac{2\gamma g}{\sqrt{6}\,\tau_0\Delta\zeta}\max_\eta\left\{\left(\frac{L^4}{l^2} + l^2 + L^2\right)\sin^2\frac{\eta}{l} + 12L^2\cos^2\frac{\eta}{l}\right\}^{1/2} \tag{4.4}$$

If $L^4 + l^4 - 11L^2l^2 > 0$, the greatest value in (4.4) is reached at $\eta = \pi l/2$.
If $L^4 + l^4 - 11L^2l^2 < 0$, the greatest value in (4.4) will be reached for $\eta = 0$ or $\eta = \pi l$.
We note that the stiffness condition depends on L and for sufficiently large L the covering generally loses stiffness.
Formula (4.3) shows that there exists a single form of covering for which c_0 is independent of L. For this it is necessary that

$$(y''/k^2(\eta)) - y = \text{const} \tag{4.5}$$

Equation (4.5) is easily integrated and there is obtained from it that the generator of such a covering is a catenary.

5. Nonalgebraicized problems. The problems considered do not

cover the whole domain of applicability of the shell approximation. Namely, such kinematic constraints as result in overdefined problems for the system (3.1) can be imposed on the allowable field $\mathbf{u}$. In this case the system (3.1) reduces to (3.2), but the $\varepsilon_0(\xi,\eta)$ in (3.2) belong to some subspace of the space of smooth vector fields, and hence, the theorem on the form of the norm of a linear functional in $L_1(D)$ is not directly applicable, which does not afford the possibility of algebraicizing the problem. However, an algorithm can be given to find a system of lower bounds for c_0. This algorithm is based on the method of Lagrange multipliers. The following theorem of Nikol'skii [5] plays an essential role here. Let a linear, continuous functional $F(\mathbf{u})$ be defined on a subspace M of the Banach space B, extracted by a finite system of linear continuous functional T_i, $T_i(\mathbf{u}) = 0$, $i = 1,\ldots, N$. Then

$$\| F \|_M = \sup_{\mathbf{u}\in M} \frac{F(\mathbf{u})}{\|\mathbf{u}\|_B} = \inf_{\lambda_i} \sup_{\mathbf{u}\in B} \left\{\left[F(\mathbf{u}) - \sum_{i=1}^{N} \lambda_i T_i(\mathbf{u})\right] \Big/ \|\mathbf{u}\|_B\right\}$$

Let us return to Example 3 in Sect. 4. Let a cylindrical shell of length L, $0 \leqslant \xi \leqslant L, \Delta\zeta = 1$ with circular cross section of radius a be subjected to the normal pressure $P = 0$, $Q = 0$, $R = R(\xi)$. If the shell is fixed only at the edge $\xi = 0$, then as follows from (3.4), (4.2), the quantity $R(\xi)$, at which the shell remains rigid should be such that

$$\max_{\xi} | R(\xi) | \leqslant \sqrt{6}\tau_0 / (2a)$$

Now we examine the same problem but under the conditions $u(0) = u(L) = 0$. The constraints mentioned result in the following condition for $\varepsilon_{\xi\xi}^0$:

$$\int_0^L \varepsilon_{\xi\xi}^\circ(\tau)\, d\tau = 0$$

Using the Nikol'skii theorem, we obtain for c_0

$$\frac{1}{c_0} = \min_{\lambda} \max_{\xi} \left\{ \frac{2}{\sqrt{6}\,\tau_0} [\lambda^2 + R^2(\xi)\, a^2 - R(\xi)\, a\lambda]^{1/2} \right\}$$

Let $R(\xi)$ be enclosed within the limits

$$R_* = \min R(\xi) \leqslant R(\xi) \leqslant \max R(\xi) = R^*$$

We examine three domains on the plane (R_*, R^*)

$$D_1 = \{R^* \geqslant R_*,\ R^* \geqslant -2R_*\}$$
$$D_2 = \{R^* \geqslant R_*,\ -\tfrac{1}{2} R_* \leqslant R^* \leqslant -2R_*\}$$
$$D_3 = \{R^* \geqslant R_*,\ R^* \leqslant -\tfrac{1}{2}R_*\}$$

Then the conditions on $R(\xi)$, for which the shell remains rigid are the following

$$R^* < \sqrt{2}\tau_0 / a \qquad \text{in } D_1$$
$$(R_*^2 + R^{*2} + R_* R^*)^{1/2} < \sqrt{6}\tau_0 / (2a) \qquad \text{in } D_2$$
$$R_* > -\sqrt{2}\,\tau_0 / a \qquad \text{in } D_3$$

We note that fixing the second edge of the shell results in an increase in its rigidity.

Let us consider the problem of the rigidity of a covering of cylindrical shape of constant thickness $\Delta\zeta = 1$. We assume that overdetermined conditions $u(0, \eta) = u(L, \eta) = 0$ relative to the system (3.1) are imposed on the field $\mathbf{u}$. This assumption results in the constraints

$$\int_0^L \varepsilon_{\xi\xi}^\circ(\tau, \eta)\, d\tau = 0, \quad \int_0^L [2\varepsilon_{\xi\eta}^\circ - (L - \tau)\, \partial\varepsilon_{\xi\xi}^\circ / \partial\eta]\, d\tau = 0$$

To solve the overdefined problem obtained, two functional Lagrange multipliers $\lambda(\eta)$, $\mu(\eta)$ must be introduced. In the problem under consideration, the number c_0 has the following form in the case of the Mises potential

$$\frac{1}{c_0} = \frac{2\gamma g}{\sqrt{6}\,\tau_0} \inf_{\lambda,\mu}\sup_{\xi,\eta} \Bigg\{\Big[\frac{(L-\xi)^2}{2}\frac{\partial^2}{\partial\eta^2}\Big(\frac{y''}{K^2} - y\Big) + \lambda(\eta) + (L-\xi)\mu'(\eta)\Big]^2\Big(\frac{y''}{K^2}\Big)^2 - \frac{y''}{K^2}\Big[\frac{(L-\xi)^2}{2}\frac{\partial^2}{\partial\eta^2}\Big(\frac{y''}{K^2} - y\Big) + \lambda(\eta) + (L-\xi)\mu'(\eta)\Big] + 3\Big[(L-\xi)\frac{\partial}{\partial\eta}\Big(\frac{y''}{K^2} - y\Big) + \mu(\eta)\Big]^2\Bigg\}^{1/2} \qquad (5.1)$$

For example, if the generator of the cylinder is a catenary, then (5.1) becomes

$$\frac{1}{c_0} \frac{2\gamma g}{\tau_0\sqrt{6}} \inf_{\lambda,\mu}\sup_{\xi,\eta}\Big\{\frac{3}{4}\Big(\frac{y''}{K^2}\Big)^2 + \Big[\lambda(\eta) + (L-\xi)\mu' - \frac{1}{2}\frac{y''}{K^2}\Big]^2 + 3\mu^2\Big\}^{1/2}$$

Evidently λ and μ should be selected as follows

$$\mu = 0, \quad \lambda = {}^1\!/_2\; y'' / K^2$$

The extremal field which satisfies the overdefined conditions is found easily in the case under consideration.

6. Proof of the asymptotic accuracy of the shell approximation for external loads of a special kind.

The asymptotic accuracy of the shell approximation is investigated under the following assumptions relative to the formulation of the problem. Let $\zeta_1 = \text{const}$, $\zeta_2 = \text{const}$, φ be independent of ζ and $\varphi(\xi, \eta, e) \geqslant \varphi(\xi, \eta, \varepsilon)$. Moreover, it is assumed that the boundary conditions are independent of ζ.

Let us consider a system of external forces of a special kind

$$F_h : (f(\xi, \eta), 0, 0)$$

Let c_h^1 denote the following number:

$$\frac{1}{c_h^1} = \sup_{\mathbf{u}(\xi,\eta,\zeta)} \Big\{\int_T \mathbf{f}\mathbf{u}\Delta d\xi\, d\eta\, d\zeta \Big[\int_T \varphi(\xi, \eta, \mathbf{e})\, \Delta d\xi\, d\eta\, d\zeta\Big]^{-1}\Big\}$$

Here the upper bound is taken over fields $\mathbf{u}$, satisfying the boundary conditions but not generally satisfying the condition (1.3). Henceforth in addition to the variables ξ, η, ζ it will be convenient to consider the variables ξ, η, σ, where $\sigma = h\zeta$, $\Sigma = \{\sigma : \sigma_1 \leqslant \sigma \leqslant \sigma_2\}$. We note that the coefficients in (1.2) are independent of h in these variables. The following lemma will be used below.

Lemma 6.1. Let $M(\xi, \eta, \sigma)$ be a sufficiently smooth function in the neighbourhood $\partial D \times \Sigma$ (∂D is a piecewise-smooth boundary of D) and $A(\xi, \eta, \sigma)$, $B(\xi, \eta, \sigma)$ sufficiently smooth functions in the same domain which are not zero on $\partial D \times \Sigma$. There exist smooth functions u_δ, v_δ, which vanish on $\partial D \times \Sigma$ and outside a boundary layer of width δ (the boundary layer enters into the domain of definition of the functions M, A, B), such that

$$\left|\frac{\partial u_\delta}{\partial \xi}\right| + \left|\frac{\partial v_\delta}{\partial \xi}\right| + \left|\frac{\partial u_\delta}{\partial \eta}\right| + \left|\frac{\partial v_\delta}{\partial \eta}\right| + \left|\frac{\partial u_\delta}{\partial \sigma}\right| + \left|\frac{\partial v_\delta}{\partial \sigma}\right| < C, \quad |u_\delta| + |v_\delta| < C\delta$$

$$\left[A\frac{\partial u_\delta}{\partial \xi} + B\frac{\partial v_\delta}{\partial \eta}\right]\Big|_{\partial D \times \Sigma} = M\,|_{\partial D \times \Sigma}$$

The lemma can be proved by direct construction of the functions u_δ, v_δ according to the scheme elucidated in [6]. Namely, it is possible to set $v_\delta = 0$, in a volume adjoining the piece of the boundary $\partial D \times \Sigma$, to which the lines $\eta = \text{const}$, $\sigma = \text{const}$ are transversal, and u_δ can be found in this volume from the conditions of the lemma. Furthermore, these local solutions are matched by using the partition of one.

Theorem 6.1. Let

$$c_h^1 \geqslant c_0 + \alpha_1(h), \quad \alpha_1(h) \to 0 \quad \text{for} \quad h \to 0 \tag{6.1}$$

Then

$$c_h = c_0 + \alpha_2(h), \quad \alpha_2(h) \to 0 \quad \text{for} \quad h \to 0$$

Proof. From the assumption relative to φ it follows $c_h \geqslant c_h^1$. Therefore, to prove the theorem it is sufficient to show that

$$c_0 + \gamma(h) \geqslant c_h, \quad \gamma(h) \to 0 \quad \text{for} \quad h \to 0 \tag{6.2}$$

Let $\mathbf{u}_t^*(\xi, \eta)$ be a family of infinitely differentiable fields on which the upper bound in (2.2) is realized. We assume that the $\mathbf{u}_t^*$ satisfy the normalization condition

$$\int_D \varphi(\xi, \eta, \varepsilon_t^{\circ *})\,|\rho_\xi|\,|\rho_\eta|\,d\xi\,d\eta = 1 \tag{6.3}$$

Here $\varepsilon_t^{\circ *}$ is the tensor ε°, evaluated by means of $\mathbf{u}_t^*$. Since the w_t^* enter algebraically in the expression for $\varepsilon_t^{\circ *}$, the w_t^* can be considered finite in D. Let $\delta(t) \to 0$ as $t \to 0$. We introduce the functions $u_{\delta(t)}(\xi, \eta)$, $v_{\delta(t)}(\xi, \eta)$, such that

$$u_{\delta(t)}|_{\partial D \times \Sigma} = v_{\delta(t)}|_{\partial D \times \Sigma} = 0 \tag{6.4}$$

$$\left(\frac{1}{|\rho_\xi|}\frac{\partial u_{\delta(t)}}{\partial \xi} + \frac{1}{|\rho_\eta|}\frac{\partial v_{\delta(t)}}{\partial \eta}\right)\Big|_{\partial D \times \Sigma} = -\left[\frac{1}{|\rho_\xi|}\frac{\partial u_t^*}{\partial \xi} + \frac{1}{|\rho_\eta|}\frac{\partial v_t^*}{\partial \eta} + \frac{1}{|\rho_\xi||\rho_\eta|}\left(u_t^*\frac{\partial |\rho_\eta|}{\partial \xi} + v_t^*\frac{\partial |\rho_\xi|}{\partial \eta}\right)\right]\Big|_{\partial D \times \Sigma}$$

By virtue of Lemma 6.1, the functions $u_{\delta(t)}$, $v_{\delta(t)}$ and $\delta(t)$ can be selected such that the relationship

$$\frac{1}{c_0} = \lim_{t\to 0}\left\{\int_D \mathbf{f}\mathbf{u}_t\,|\rho_\xi|\,|\rho_\eta|\,d\xi\,d\eta\left[\int_D \varphi(\xi,\eta,\varepsilon_t^\circ)\,|\rho_\xi|\,|\rho_\eta|\,d\xi\,d\eta\right]^{-1}\right\}$$

will be satisfied for the vector functions $u_t : u_t = u_t^* + u_{\delta(t)}$, $v_t = v_t^* + v_{\delta(t)}$, $w_t = w_t^*$ where $\varepsilon_{\xi\xi t}^\circ$, $\varepsilon_{\xi\eta t}^\circ$, $\varepsilon_{\eta\eta t}^\circ$ are evaluated by means of $\mathbf{u}_t = (u_t, v_t, w_t)$. It follows from conditions (6.4) that:

$$\overset{\bullet}{\varepsilon}_{\xi\xi t} + \overset{\circ}{\varepsilon}_{\eta\eta t}\,|_{\partial D\times\Sigma} = 0 \tag{6.5}$$

The vector fields u_t satisfy the same boundary conditions as does $\mathbf{u}_t^*$. To obtain (6.2), let us examine $\mathbf{u}_{th}$

$$u_{th} = u_t(\xi,\eta) + \zeta h\,[u_t^1(\xi,\eta) + u_t^2(\xi,\eta,\zeta h)]$$
$$v_{th} = v_t(\xi,\eta) + \zeta h\,[v_t^1(\xi,\eta) + v_t^2(\xi,\eta,\zeta h)]$$
$$w_{th} = w_t(\xi,\eta) + h w_{th}^1(\xi,\eta,\zeta)$$

Here

$$u_t^1 = -\frac{1}{|\rho_\xi|}\frac{\partial w_t}{\partial\xi} - K_\xi u_t, \quad v_t^1 = -\frac{1}{|\rho_\eta|}\frac{\partial w_t}{\partial\eta} - K_\eta v_t$$

The functions $u_t^2(\xi,\eta,\sigma)$, $v_t^2(\xi,\eta,\sigma)$ equal zero on $\partial D\times\Sigma$ and

$$\left|\frac{\partial u_t^2}{\partial\xi}\right| + \left|\frac{\partial v_t^2}{\partial\xi}\right| + \left|\frac{\partial u_t^2}{\partial\eta}\right| + \left|\frac{\partial v_t^2}{\partial\eta}\right| + \left|\frac{\partial u_t^2}{\partial\sigma}\right| + \left|\frac{\partial v_t^2}{\partial\sigma}\right| < C_t$$

$$\left(\frac{1}{H_\xi}\frac{\partial u_t^2}{\partial\xi} + \frac{1}{H_\eta}\frac{\partial v_t^2}{\partial\eta}\right)\Big|_{\partial D\times\Sigma} = \left\{u_t\left[\frac{1}{H_\xi}\frac{\partial K_\xi}{\partial\xi} - \frac{1}{\sigma}\left(\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial\xi} - \frac{1}{|\rho_\xi||\rho_\eta|}\frac{\partial|\rho_\eta|}{\partial\xi}\right)\right] + v_t\left[\frac{1}{H_\eta}\frac{\partial K_\eta}{\partial\eta} - \frac{1}{\sigma}\left(\frac{1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial\eta} - \frac{1}{|\rho_\xi||\rho_\eta|}\frac{\partial|\rho_\xi|}{\partial\eta}\right)\right]\right\}\Big|_{\partial D\times\Sigma}$$

The existence of u_t^2, v_t^2 with the properties mentioned follows from Lemma (6.1). The function w_{th}^1 is determined from the equations

$$-\frac{\partial w_{th}^1}{\partial\zeta} + h w_{th}^1\left(\frac{K_\xi}{1-h\zeta K_\xi} + \frac{K_\eta}{1-h\zeta K_\eta}\right) = w_t\left(\frac{K_\xi}{1-h\zeta K_\xi} + \frac{K_\eta}{1-h\zeta K_\eta}\right) + \frac{1}{|\rho_\xi|}\frac{\partial u_t}{\partial\xi} + \frac{v_t}{|\rho_\xi||\rho_\eta|}\frac{\partial|\rho_\xi|}{\partial\eta} + \frac{1}{|\rho_\eta|}\frac{\partial v_t}{\partial\eta} + \frac{u_t}{|\rho_\xi||\rho_\eta|}\frac{\partial|\rho_\eta|}{\partial\xi} + \tag{6.6}$$

$$h\zeta\Big[\frac{K_\xi}{H_\xi}\frac{\partial u_t}{\partial\xi}+\frac{K_\eta}{H_\eta}\frac{\partial v_t}{\partial\eta}+\frac{u_t}{h\zeta}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial\xi}-\frac{1}{|\rho_\xi||\rho_\eta|}\frac{\partial|\rho_\eta|}{\partial\xi}\Big)+\frac{v_t}{h\zeta}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial\eta}-\frac{1}{|\rho_\xi||\rho_\eta|}\frac{\partial|\rho_\xi|}{\partial\eta}\Big)+$$
$$\frac{1}{H_\xi}\frac{\partial u_t^1}{\partial\xi}+\frac{v_t^1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial\eta}+\frac{1}{H_\eta}\frac{\partial v_t^1}{\partial\eta}+\frac{u_t^1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial\xi}+$$
$$\frac{1}{H_\xi}\frac{\partial u_t^2}{\partial\xi}+\frac{v_t^2}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial\eta}+\frac{1}{H_\eta}\frac{\partial v_t^2}{\partial\eta}+\frac{u_t^2}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial\xi}\Big]$$

It follows from (6.4)-(6.6) that w_{th}^1 can be selected to vanish on $\partial D\times\Sigma$, where $\mathbf{u}_{th}$ satisfies the same boundary conditions as does $\mathbf{u}_t^*$, and satisfies the incompressibility condition.

The following inequality results from (1.4)

$$\frac{1}{c_h}\geqslant\int_T \mathbf{f}\mathbf{u}_{th}\Delta d\xi\,d\eta\,d\zeta\Big[\int_T\varphi(\xi,\eta,\mathbf{e}_{th})\,\Delta d\xi\,d\eta\,d\zeta\Big]^{-1}\geqslant$$
$$\Big[\int_D \mathbf{f}\mathbf{u}_t^*|\rho_\xi||\rho_\eta|\,d\xi\,d\eta+hE(t)\Big]\times$$
$$\Big[\int_D\varphi(\xi,\eta,\mathbf{e}_t^{0*})|\rho_\xi||\rho_\eta|\,d\xi\,d\eta+hG(t)\Big]^{-1} \qquad (6.7)$$

where $E(t)$, $G(t)$, generally tend to infinity as $t\to 0$. Let us select the dependence $t(h)$ so that $hE(t(h))\to 0$, $G(t(h))\,h\to 0$ as $h\to 0$. Then (6.2) follows from (6.3), (6.7), where $\gamma(h)$ can be estimated in terms of the given problem. The theorem is proved.

We note that finding the quantity c_h^1 is a more simple problem than finding c_h, where the inequality $c_h^1\leqslant c_h$ always holds.

Let us indicate certain cases when (6.1) is satisfied and consider

$$1/c_0(\sigma)=\sup_{\mathbf{u}(\xi,\eta)}\{A(\sigma)B^{-1}(\sigma)\}$$
$$A(\sigma)=\int_D \mathbf{f}\mathbf{u}\Delta(\sigma)\,d\xi\,d\eta,\quad B(\sigma)=\int_D\varphi(\xi,\eta,\mathbf{e})\,\Delta(\sigma)\,d\xi\,d\eta$$
$$\Delta(\sigma)=\Delta,\quad \zeta_1 h=\sigma_1\leqslant\sigma\leqslant\sigma_2=\zeta_2 h$$

Theorem 6.2.

$$c_h^1\geqslant\inf_\sigma c_0(\sigma)$$

Proof. The assertion in the theorem is equivalent to the inequality

$$1/c_h^1\leqslant\sup_\sigma\{1/c_0(\sigma)\}$$

Let us prove this inequality. In fact

$$\frac{1}{c_h^1}=\sup_{\mathbf{u}(\xi,\eta,\zeta)}\Big\{\int_{\zeta_1}^{\zeta_2}A(\zeta h)\,d\zeta\Big[\int_{\zeta_1}^{\zeta_2}B(h\zeta)\,d\zeta\Big]^{-1}\Big\}\leqslant$$

$$\sup_{\sigma_1\leqslant\sigma\leqslant\sigma_2}\ \sup_{u(\xi,\eta)}\{A(\sigma)B^{-1}(\sigma)\}$$

The assumption that the boundary conditions for $\mathbf{u}(\xi, \eta, \zeta)$ are independent of ζ is used in obtaining the last inequality.

Therefore, to establish (6.1) it is sufficient to show that

$$c_0(\sigma)\to c_0 \quad \text{for} \quad h\to 0 \tag{6.8}$$

The relationship (6.8) can be obtained, for instance, for surfaces of positive or zero Gaussian curvature. In this latter case, formulas for the general solution of a system of equations of infinitesimal flexure of developable surfaces (3.1) can be used [3]. We present still another class of problems when (6.8) holds. For the system

$$\begin{gathered}\frac{1}{H_\xi}\frac{\partial u}{\partial \xi}+\frac{v}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial \eta}-\frac{wK_\xi}{1-h\zeta K_\xi}=e_{\xi\xi}\\ \frac{1}{2}\left[\frac{H_\xi}{H_\eta}\frac{\partial}{\partial\eta}\left(\frac{u}{H_\xi}\right)+\frac{H_\eta}{H_\xi}\frac{\partial}{\partial\xi}\left(\frac{v}{H_\eta}\right)\right]=e_{\xi\eta}\\ \frac{1}{H_\eta}\frac{\partial v}{\partial \eta}+\frac{u}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi}-\frac{wK_\eta}{1-h\zeta K_\eta}=e_{\eta\eta}\end{gathered} \tag{6.9}$$

let there be a homogeneous boundary value problem, uniquely solvable for any sufficiently smooth $\boldsymbol{\varepsilon}=(e_{\xi\xi}, e_{\xi\eta}, e_{\eta\eta})$ whose inverse operator will be denoted by $G(\sigma):\mathbf{u}=G(\sigma)\,\boldsymbol{\varepsilon}$. Let $G^*(\sigma)$ be the operator conjugate to $G(\sigma)$, i.e.,

$$\int_D \Delta(\sigma)\,\boldsymbol{\varepsilon}G^*(\sigma)\,\mathbf{f}d\xi\, d\eta=\int_D\Delta(\sigma)\,\mathbf{f}G(\sigma)\,\boldsymbol{\varepsilon}d\xi\, d\eta$$

Then (the quantity β has been defined in Sect. 1)

$$\begin{gathered}\frac{1}{c_0(\sigma)}\leqslant\left[c_0(\sigma_1)\min_{\xi,\eta}\frac{\Delta(\sigma)}{\Delta(\sigma_1)}\right]^{-1}+\\ [\beta\min_{\xi,\eta}\Delta(\sigma)]^{-1}\max_{\xi,\eta}|(\Delta(\sigma)G^*(\sigma)-\Delta(\sigma_1)G^*(\sigma_1))f|^*\\ |a|^*=\left(a_{\xi\xi}^2+\frac{1}{2}a_{\xi\eta}^2+a_{\eta\eta}^2\right)^{1/2},\ \text{если}\ a=(a_{\xi\xi}, a_{\xi\eta}, a_{\eta\eta})\end{gathered} \tag{6.10}$$

Therefore, if

$$\max_{\xi,\eta}|(G^*(\sigma)-G^*(\sigma_1))\,\mathbf{f}\,|^*\to 0 \quad \text{for} \quad h\to 0 \tag{6.11}$$

then (6.8) follows from (6.10). We emphasize that (6.11) should be satisfied for fixed **f**.

7. Proof of the asymptotic accuracy of the shell approximation for external loads of general form.

Let us assume that the boundary conditions for shell supports generally form an overdetermined problem for (6.9), where we obtain a uniquely solvable inverse problem for $G(\sigma) : \mathbf{u} = G(\sigma)\,\mathbf{f}$ by removing certain conditions from this overdetermined problem. The overdetermined conditions imposed on $\mathbf{u}$ result in $\boldsymbol{\varepsilon}$ forming a certain linear subspace in the space of smooth vector-functions. Let us still assume that $G^*(\sigma)$ is a continuous operator from $C^k(D \times \Sigma) \to C(D \times \Sigma)$.

We first examine a field of external force of the following kind:

$$F_h^i = (\mathbf{f}_i(\xi, \eta, \zeta, h), 0, 0), \quad i = 1, 2$$

Let $c_i(h)$ be the limit load coefficients corresponding to these force fields.

Theorem 7.1. If

$$\| \mathbf{f}_1 - \mathbf{f}_2 \|_{C^k(D\times\Sigma)} \leqslant \alpha(h), \quad \alpha(h) \to 0 \quad \text{for } h \to 0$$

then there exists a $\gamma(h) \to 0$ as $h \to 0$ such that $| c_2(h) - c_1(h) | \leqslant \gamma(h)$.

Proof. The assertion in the theorem follows directly from the inequality

$$\sup_{\mathbf{u},\, \operatorname{div}\mathbf{u}=0} \left\{ I^{-1}(e) \int_D \int_{\zeta_1}^{\zeta_2} [\mathbf{f}_1 - \mathbf{f}_2]\, \mathbf{u} \Delta d\xi\, d\eta\, d\zeta \right\} \leqslant$$

$$\sup_{\varepsilon(\sigma)} \left\{ I^{-1}(\varepsilon) \int_D \int_{\zeta_1}^{\zeta_2} \Delta [\mathbf{f}_1 - \mathbf{f}_2]\, G(\sigma)\, \varepsilon d\xi\, d\eta\, d\zeta \right\} =$$

$$\sup_{\varepsilon(\sigma)} \left\{ I^{-1}(\varepsilon) \int_D \int_{\zeta_1}^{\zeta_2} \Delta (G^*(\sigma)[\mathbf{f}_1 - \mathbf{f}_2])\, \varepsilon d\xi\, d\eta\, d\zeta \right\}$$

$$I(e) = \int_D \int_{\zeta_1}^{\zeta_2} \varphi(\xi, \eta, e)\, \Delta d\xi\, d\eta\, d\zeta$$

Let us turn to the reduction of the external forces of a general kind to the forces of a special kind examined in Sect. 6., and consider the transformation of the following integral

$$\int_D f_\zeta^i(\xi, \eta, h)\, w(\xi, \eta, \zeta_i)\, \Delta_i d\xi\, d\eta = \tag{7.1}$$

$$\int_D \int_{\zeta_1}^{\zeta_2} \left[\frac{f_\zeta^i(\xi, \eta, h)}{\zeta_2 - \zeta_1} \Delta_i \left(w(\xi, \eta, \zeta) - \int_{\zeta_i}^{\zeta} \frac{\partial w(\xi, \eta, \lambda)}{\partial \lambda} d\lambda \right) \right] d\xi\, d\eta\, d\zeta =$$

$$\int_D \int_{\zeta_1}^{\zeta_2} \frac{f_\zeta^i(\xi, \eta, h)}{\zeta_2 - \zeta_1} \frac{\Delta_i}{\Delta} \Delta w(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta + \int_D \int_{\zeta_1}^{\zeta_2} h P_\zeta^i e_{\zeta\zeta} \Delta d\xi\, d\eta\, d\zeta$$

We note that P_ζ^i is a bounded function of its arguments.

Now, let us consider the integral

$$\int_D \int_{\zeta_1}^{\zeta_2} f_\zeta(\xi, \eta, \zeta, h)\, w(\xi, \eta, \zeta)\, \Delta d\xi\, d\eta\, d\zeta = \int_D \int_{\zeta_1}^{\zeta_2} f_\zeta \Big[w(\xi, \eta, \zeta_1) +$$

$$\int_{\zeta_1}^{\zeta} \frac{\partial w(\xi,\eta,\lambda)}{\partial\lambda} d\lambda \Big] \Delta d\xi\, d\eta\, d\zeta = \int_D \int_{\zeta_1}^{\zeta_2} h P_\zeta{}^3 e_{\zeta\zeta} \Delta d\xi\, d\eta\, d\zeta + A$$

$$A = \int_D f_\zeta{}^3(\xi,\eta,\zeta)\, w(\xi,\eta,\zeta_1)\, d\xi\, d\eta$$

The function $P_\zeta{}^3$ is a bounded function of its arguments. Furthermore, the integral A is transformed by means of (7.1).
Let us turn to a consideration of the following integrals

$$A_i(u) = \int_D f_\xi{}^i(\xi,\eta,h)\, u(\xi,\eta,\zeta_i)\, \Delta_i d\xi\, d\eta = \int_D \int_{\zeta_1}^{\zeta_2} \frac{f_\xi{}^i H_\xi \Delta_i}{\zeta_2-\zeta_1} \times \tag{7.2}$$

$$\Big[\frac{u(\xi,\eta,\zeta)}{H_\xi} - \int_{\zeta_i}^{\zeta} \frac{\partial}{\partial\lambda}\Big(\frac{u(\xi,\eta,\lambda)}{H_\xi(\xi,\eta,\lambda)}\Big) d\lambda\Big] d\xi\, d\eta\, d\zeta =$$

$$\int_D \int_{\zeta_1}^{\zeta_2} \Big[\frac{f_\xi{}^i \Delta_i}{\zeta_2-\zeta_1} u(\xi,\eta,\zeta) + h\Delta P_\xi{}^i e_{\xi\zeta} + h\Delta Q_\xi{}^i \frac{\partial w}{\partial\xi}\Big] d\xi\, d\eta\, d\zeta$$

Under the assumption of sufficient smoothness of $f_\xi{}^i$ the functions $P_\xi{}^i$, $Q_\xi{}^i$ are also sufficiently smooth functions of their arguments. Let $f_\xi{}^i$, $f_\eta{}^i$ vanish on those parts of $\partial D \times \Sigma$, where $w(\xi,\eta,\zeta)$ can be different from zero. Then (7.2) can be rewritten as

$$A_i(u) = \int_D \int_{\zeta_1}^{\zeta_2} \frac{f_\xi{}^i \Delta_i}{\zeta_2-\zeta_1} u(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta + \tag{7.3}$$

$$h \int_D \int_{\zeta_1}^{\zeta_2} \Delta P_\xi{}^i e_{\xi\zeta} d\xi\, d\eta\, d\zeta + h \int_D \int_{\zeta_1}^{\zeta_2} \Delta R_\xi{}^i w d\xi\, d\eta\, d\zeta, \quad i = 1, 2$$

Repeating the previous manipulations, a formula can be obtained for $A_i(v)$, which is analogous to (7.3).
Moreover, let us examine the integral

$$A(u) = \int_D \int_{\zeta_1}^{\zeta_2} f_\xi(\xi,\eta,\zeta,h)\, u\Delta d\xi\, d\eta\, d\zeta = \int_D \int_{\zeta_1}^{\zeta_2} \Delta H_\xi f_\xi \Big[\frac{u(\xi,\eta,\zeta_1)}{H_\xi(\xi,\eta,\zeta_1)} + \tag{7.4}$$

$$\int_{\zeta_1}^{\zeta} \frac{\partial}{\partial\lambda}\Big(\frac{u(\xi,\eta,\lambda)}{H_\xi(\xi,\eta,\lambda)}\Big) d\lambda\Big] d\xi\, d\eta\, d\zeta = \int_D \int_{\zeta_1}^{\zeta} \frac{1}{\zeta_2-\zeta_1}\Big(\int_{\zeta_1}^{\zeta_2} f_\xi d\zeta\Big) \times$$

$$u\Delta d\xi\, d\eta\, d\zeta + h \int_D \int_{\zeta_1}^{\zeta_2} P_\xi{}^3 e_{\xi\zeta} d\xi\, d\eta\, d\zeta + h \int_D \int_{\zeta_1}^{\zeta_2} R_\xi{}^3 w d\xi\, d\eta\, d\zeta$$

A formula analogous to (7.4) can be obtained for the integral $A(v)$.
Thus, a functional corresponding to the power of external forces of a general kind is converted into a functional corresponding to the power of external forces examined in Sect. 6 and the functional

$$h \int_D \int_{\zeta_1}^{\zeta_2} a(\xi, \eta, h)\, w(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \tag{7.5}$$

where $a(\xi, \eta, h)$ is a function of the order of one in h. The shell approximation will be asymptotically exact if (7.5) yields a small contribution to c_h. The condition of smallness of this contribution can be investigated on the basis of properties of solutions of boundary value problems for the system (6.9). A case is possible when (7.5) will influence c_h substantially and the shell approximation will lead to an incorrect result.
As an illustration, we consider the plate

$$K_\xi = K_\eta = 0, \quad (\xi, \eta) \in D, \quad -1 \leqslant \zeta \leqslant 1$$

In the case of a plate the shell approximation agrees with the approximation of the plane state of stress [7].
Let the boundary conditions have the form $u|_{\partial D \times \Sigma} = 0$ and let surface forces with density $h(f(\xi, \eta), 0, 0)$ act on the plane $\zeta = 1$. In this case (2.2) has the form

$$\frac{1}{c_0} = \sup_{u(\xi, \eta)} \left\{ \Phi_0^{-1} \int_D f u\, d\xi\, d\eta \right\}$$

$$\Phi_0 = 2 \int_D \varphi\left(\xi, \eta, \frac{\partial u}{\partial \xi}, \frac{1}{2}\left(\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi}\right) \frac{\partial v}{\partial \eta}, -\frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \eta}, 0, 0\right) d\xi\, d\eta$$

If f is a continuous function, then $c_0 > 0$. Let us turn to the formula for c_h. Let us transform the integral

$$\int_D f u(\xi, \eta, 1)\, d\xi\, d\eta =$$

$$\frac{1}{2} \int_D \int_{-1}^{1} f \left[u(\xi, \eta, \zeta) + \int_\zeta^1 \frac{\partial u(\xi, \eta, \lambda)}{\partial \lambda} d\lambda \right] d\xi\, d\eta\, d\zeta = J(\mathbf{u})$$

$$J(\mathbf{u}) = \int_T \left[\frac{1}{2} f u + h f (1 + \zeta)\, e_{\zeta\xi} + \frac{h}{2} f_\xi' (1 + \zeta)\, w \right] d\xi\, d\eta\, d\zeta$$

Then

$$\frac{1}{c_h} = \sup_{\mathbf{u},\, \operatorname{div} \mathbf{u} = 0} \{ I^{-1}(e)\, J(\mathbf{u}) \}$$

We set

$$u = u_0(\xi, \eta) - \zeta\frac{\partial q}{\partial \xi}, \quad v = v_0(\xi, \eta) - \zeta\frac{\partial q}{\partial \eta}, \quad q = q(\xi, \eta)$$

$$w = \frac{1}{h}q - h\zeta\frac{\partial u_0}{\partial \xi} - h\zeta\frac{\partial v_0}{\partial \eta} + \frac{h^2\zeta^2}{2}\Delta q, \quad \Delta q = \frac{\partial^2 q}{\partial \xi^2} + \frac{\partial^2 q}{\partial \eta^2}$$

We find from the formula for c_h

$$\lim_{h\to 0}\frac{1}{c_h} \geqslant \sup_{u_0, q}\left\{\Phi_1^{-1}\left[\int_D f u_0\, d\xi\, d\eta + \int_D f_\xi' q\, d\xi\, d\eta\right] = \frac{1}{c^*}\right. \tag{7.6}$$

$$\Phi_1 = 2\int_D \varphi\left(\xi, \eta, \frac{\partial u_0}{\partial \xi} - \zeta\frac{\partial^2 q}{\partial \xi^2}, \frac{1}{2}\left(\frac{\partial u_0}{\partial \eta} + \frac{\partial v_0}{\partial \xi}\right) - \zeta\frac{\partial^2 q}{\partial \xi\,\partial \eta}\right.$$

$$\left.\frac{\partial v_0}{\partial \eta} - \zeta\frac{\partial^2 q}{\partial \eta^2}, \zeta\Delta q - \frac{\partial u_0}{\partial \xi} - \frac{\partial v_0}{\partial \eta}, 0, 0\right) d\xi\, d\eta$$

Using the variational asymptotic method proposed in [8], we can show that the equality holds in (7.6). If $f_\xi' = 0$, then as follows from the results in Sect. 7, the shell approximation is asymptotically exact. However, if $f_\xi' \neq 0$, then $c^* < c_0$ and the shell approximation leads to an incorrect result.

In conclusion let us note that the proposed method of proving the asymptotic accuracy permits obtaining quantitative estimates of the difference between c_0 and c_h, as well as an indication of the limits of applicability of the heuristic formulas.

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